A Finite Element Procedure for Large Deflection Analysis of Plates with Initial Deflections

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The concepts of the matrix displacement approach to finite element structural analysis are extended to analyze the large deflection behavior of plates with initial deflections. The initial deflections are considered to be of the order of magnitude of the plate thickness. The large deflection phenomena are accounted for by considering the influence of the element membrane forces due to initial deflection and large bending on element flexure stiffness matrices. The effects of element membrane forces due to initial deflection and large bending are characterized in the nonlinear stiffness formulations by the zero, first, and second order incremental stiffness matrices. Representations of the nonlinear effects are formulated for a conforming rectangular plate finite element. The formulations are appropriate for both the iterative and incremental approaches. Large deflection behaviors are predicted for rectangular plates with different length-width ratios and various boundary conditions. Comparisons with some available alternative analytic solutions are provided. Examples are performed by a piecewise linear incremental approach. A reduction scheme which removes certain degrees of freedom from the problem statement is incorporated into the numerical procedure.

Nomenclature

A,B = length and width of a rectangular plate, respectively
c = size of the first load increment
D = flexural rigidity of a plate
E = modulus of elasticity
h = plate thickness
i = incremental step number
$[k]$ = element linear stiffness matrix
$[n_0],[n_1],[n_2]$ = zero, first, and second order element incremental stiffness matrix, respectively
p = uniformly distributed lateral load
$\{p\}$ = vector of element nodal loads
$\{q_o\},\{q\}$ = the vectors of element initial nodal displacements and net nodal displacements, respectively
r = geometric constant shown in Eq. (24)
U = strain energy of a plate element
w_o,w = initial and net deflection of a plate element
$\bar{w}_o,\bar{w}$ = initial and net deflection of the whole plate
W_o = maximum initial deflection of the whole plate
δ = variational operator
Δ = incremental operator
ν = Poisson's ratio
$\{\ \}$ = column vector
$\lfloor\ \rfloor$ = row vector
$[\ ]$ = rectangular matrix

I. Introduction

If the deflection of a plate is of the order of magnitude of its thickness but is still small relative to other dimensions, the analysis of the problem should include the strain of the middle plane of the plate. Classical formulation of this problem leads to a set of nonlinear partial differential equations which are characterized by the coupling of the dependent variables describing the membrane and bending behavior of the plate. These equations are difficult to solve.

In many technical fields of aircraft construction, shipbuilding, and instrument manufacturing, plates are finding use with large deflections. During the fabricating process, the plates usually inherit initial curvatures. The analyses are more complicated than those for ideally flat plates.

An alternative approach to such problems is now available in the form of the finite element method. This method formulates the problem in matrix form and achieves the numerical solutions without directly solving the differential equations.

The objective here is to develop the nonlinear stiffness matrix formulations which are appropriate for large deflection analysis of plates with initial deflections. The initial deflections are considered to be of the order of magnitude of the plate thickness. The effects due to initial deflection and large bending on the nonlinear stiffness formulations are characterized by the zero, first, and second order incremental stiffness matrices. Representations of the nonlinear effects are formulated for a conforming rectangular plate finite element appropriate for both the iterative and incremental approaches. The formulations are developed on the basis of the von Karman's assumptions of large deflection of plate and by the method of minimizing potential energy. The use of the principle of minimum potential energy to formulate the incremental stiffness matrix has been given in Ref. 1.

Illustrative examples are performed by a step-by-step linear incremental procedure. This procedure was introduced in Ref. 2 and discussed extensively in Ref. 3. The examples include initially deflected rectangular plates with different length-width ratios and various boundary conditions, i.e., all edges simply supported, all edges clamped, two opposite edges simply supported and the other two clamped. The loads are uniformly distributed for all cases. For the case of zero initial deflection, the results are compared with available alternative solutions[4–8] and excellent agreements are found. For plates with initial deflections, an approximate solution is available for a simple supported long rectangular plate.[8] The solution provides convincing evaluation for the present formulations and results.

Received March 30, 1970; revision received September 18, 1970. The author wishes to acknowledge R. H. Gallagher of Cornell University for the guidance and help. He also appreciates the service of the CDC 6500 computer from Purdue Computer Science Center.

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II. Formulation

Variational Equation of Equilibrium

The total potential energy Π of a deformed plate with initial deflection of the order of magnitude of thickness and with the same order additional bending deflection, is defined as

$$\Pi = U - W \tag{1}$$

where U is the potential energy of deformation and W is the potential energy of external loading.

The state of equilibrium of a deformed plate can be characterized as that for which the first variation of the total potential energy of the system is equal to zero,

$$\delta\Pi = \delta U - \delta W = 0 \tag{2}$$

or

$$\delta U = \delta W \tag{3}$$

If the functions for U and W are formed, the equation of equilibrium can be obtained by executing the variation as indicated by Eq. (3).

The potential energy of the external load is

$$W = p_i q_i \tag{4}$$

in which the repeated indices imply summation, p_i is the external load, and q_i is the displacement.

The potential energy of deformation for a plate element with large deflection written in terms of the deflection and the strain of the middle surface is[7,9]

$$U = (D/2)\iint\{[(\nabla^2 w)^2 + (12/h^2)e_1{}^2] - 2(1-\nu)[(12/h^2)e_2 + (\partial^2 w/\partial x^2)(\partial^2 w/\partial y^2) - (\partial^2 w/\partial x \partial y)^2]\}dxdy \tag{5a}$$

in which

$$e_1 = \epsilon_x + \epsilon_y, \quad e_2 = \epsilon_x\epsilon_y - \tfrac{1}{4}\epsilon_{xy}{}^2 \tag{5b}$$

In Eq. (5b), the middle surface strains ϵ_x, ϵ_y, and ϵ_{xy} are (from p. 14 of Ref. 8)

$$\begin{aligned}\epsilon_x &= \partial u/\partial x + \tfrac{1}{2}[\partial(w+w_o)/\partial x]^2 - \tfrac{1}{2}(\partial w_o/\partial x)^2 \\ \epsilon_y &= \partial v/\partial y + \tfrac{1}{2}[\partial(w+w_o)/\partial y]^2 - \tfrac{1}{2}(\partial w_o/\partial y)^2 \\ \epsilon_{xy} &= (\partial u/\partial y + \partial v/\partial x) + [\partial(w+w_o)/\partial x] \times [\partial(w+w_o)/\partial y] - (\partial w_o/\partial x)(\partial w_o/\partial y)\end{aligned} \tag{6}$$

Corresponding to the middle surface strains, the middle surface stresses σ_x, σ_y, and σ_{xy} are

$$\begin{aligned}\sigma_x &= [E/(1-\nu^2)](\epsilon_x + \nu\epsilon_y) \\ \sigma_y &= [E/(1-\nu^2)](\epsilon_y + \nu\epsilon_x) \\ \sigma_{xy} &= [E/2(1+\nu)]\epsilon_{xy}\end{aligned} \tag{7}$$

By substituting Eq. (6) into Eqs. (5a) and (5b), the following energy expression is obtained

$$U = U_k + U_0 + U_1 + U_2 \tag{8a}$$

where

$$U_k = \frac{D}{2}\iint\left\{(\nabla^2 w)^2 + \frac{12}{h^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \frac{12}{h^2}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{3}{h^2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]\right\}dxdy \tag{8b}$$

$$U_0 = -\frac{3D}{h^2}\iint\left\{\left[\left(\frac{\partial w_o}{\partial x}\right)^2 + \nu\left(\frac{\partial w_o}{\partial y}\right)^2\right]\left[\frac{\partial(w+w_o)}{\partial x}\right]^2 + \left[\left(\frac{\partial w_o}{\partial y}\right)^2 + \nu\left(\frac{\partial w_o}{\partial x}\right)^2\right]\left[\frac{\partial(w+w_o)}{\partial y}\right]^2 + 2(1-\nu)\frac{\partial w_o}{\partial x}\frac{\partial w_o}{\partial y}\frac{\partial(w+w_o)}{\partial x}\frac{\partial(w+w_o)}{\partial y}\right\}dxdy \tag{8c}$$

$$U_1 = \frac{6D}{h^2}\iint\left\{\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right)\left[\frac{\partial(w+w_o)}{\partial x}\right]^2 + \left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right)\left[\frac{\partial(w+w_o)}{\partial y}\right]^2 + (1-\nu)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \times \frac{\partial(w+w_o)}{\partial x}\frac{\partial(w+w_o)}{\partial y}\right\}dxdy \tag{8d}$$

$$U_2 = \frac{3D}{2h^2}\iint\left\{\left[\frac{\partial(w+w_o)}{\partial x}\right]^2 + \left[\frac{\partial(w+w_o)}{\partial y}\right]^2\right\}^2 dxdy \tag{8e}$$

The zero- and first-order displacement derivative terms, such as $\frac{1}{4}(\partial w_o/\partial x)^4$, $-\partial u/\partial x(\partial w_o/\partial x)^2$... etc., do not contribute to the stiffness matrices. Therefore, they are neglected from the above energy expression. Note that the derivatives of initial deflection w_o (constant slopes) are zero-order terms.

Element Equilibrium Equation

If the displacement functions u, v, and w for a plate finite element are appropriately chosen, the potential energy of deformation for this element can be obtained in a matrix form by substituting the displacement functions into Eqs. (8). The displacement function for initial deflection w_o is the same as the displacement function w except that the nodal degrees of freedom in the former are prescribed as constants in accordance with the initial shape of the plate. The strain energy function thus obtained will be in the form

$$U = \tfrac{1}{2}\lfloor q\rfloor[k]\{q\} + \tfrac{1}{2}\lfloor q+q_o\rfloor[n_0]\{q+q_o\} + \tfrac{1}{6}\lfloor q+q_o\rfloor[n_1]\{q+q_o\} + \tfrac{1}{12}\lfloor q+q_o\rfloor[n_2]\{q+q_o\} \tag{9}$$

The coefficients in each of the above matrices can be obtained by performing the second partial differentiations of the proper component of strain energy with respect to the associated nodal displacements. For example, the coefficients in the ith row and jth column of the above matrices are obtained by

$$k_{ij} = \frac{\partial^2 U_k}{\partial q_i \partial q_j}; \quad n_{0ij} = \frac{\partial^2 U_0}{\partial q_i \partial q_j}; \quad n_{1ij} = \frac{\partial^2 U_1}{\partial q_i \partial q_j}; \quad n_{2ij} = \frac{\partial^2 U_2}{\partial q_i \partial q_j} \tag{10}$$

The potential energy of external loading expressed by Eq. (4) can now be written in a matrix form

$$W = \lfloor p\rfloor\{q\} \tag{11}$$

where $\lfloor p\rfloor$ is the row vector of external loads.

The element equilibrium equation can be obtained by executing the first variation of Eqs. (9) and (11) and following the equilibrium condition defined by Eq. (3),

$$\{p\} = [k]\{q\} + [[n_0] + \tfrac{1}{2}[n_1] + \tfrac{1}{3}[n_2]]\{q+q_o\} \tag{12}$$

It is to be noted that the matrices $[n_0]$, $[n_1]$, and $[n_2]$ are respectively the zero, first, and second-order functions of the *gross* displacement vector $\{q+q_o\}$.

Once the gridwork for a plate structure is determined, the element stiffness formulations of Eq. (12) can be assembled to form the total system stiffness equation which is represented by capital letters,

$$\{P\} = [K]\{Q\} + [[N_0] + \tfrac{1}{2}[N_1] + \tfrac{1}{3}[N_2]]\{Q+Q_o\} \tag{13}$$

This nonlinear stiffness equation is readily applicable for the direct iterative analysis.

An alternative formulation which is appropriate for a step-by-step linear incremental procedure can now be obtained by applying an incremental operator Δ to Eq. (13). In other words, replacing the displacements $\{Q+Q_o\}$ by $\{Q+\Delta Q+Q_o\}$ in Eq. (13), then subtracting the original Eq. (13) from

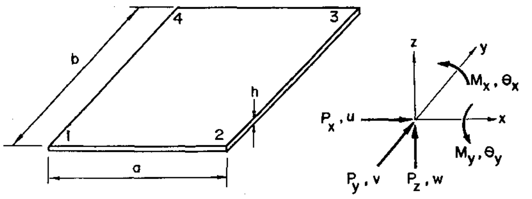

Fig. 1 Element geometry and joint force and displacement (Mxy and θ_{xy} not shown).

the replaced equation and neglecting the second and third order terms of incremental degrees of freedom, $\Delta Q_i \Delta Q_j$ and $\Delta Q_i \Delta Q_j \Delta Q_k$. With the knowledge that $[N_0]$, $[N_1]$, and $[N_2]$ are, respectively, the zero-, first-, and second-order functions of the gross displacement vector $\{Q + Q_o\}$, the incremental operation of Eq. (13) yields

$$\{\Delta P\} = [[K] + [N_0] + [N_1] + [N_2]]\{\Delta Q\} \tag{14}$$

This nonlinear matrix equations is now applicable for a step-by-step linear incremental procedure. The incremental procedure can be described by

$$\{\Delta Q\}_{i+1} = [[K] + [N_0] + [N_1] + [N_2]]_i^{-1}\{\Delta P\}_{i+1} \tag{15}$$

In step $i + 1$, the load increment is to be multiplied by the inverse of the sum of four stiffness matrices which is based on the displacement state at the end of step i.

Stiffness Formulation for a Rectangular Element

The nonlinear stiffness formulations for large deflection analysis of plates with initial curvatures are formulated for a rectangular plate finite element. The element is shown in Fig. 1 with six degrees of freedom at each nodal point. These are: two inplane displacements u and v in x and y directions, respectively; one transverse deflection w; two rotations θ_x and θ_y about y and x axes, respectively; and a generalized twist θ_{xy}.

The u and v displacement functions chosen are the so-called bilinear or first order Lagrangian interpolation functions which provide linear representation in the x and y directions. The w displacement function is obtained from the fourth order Hermitian polynomial approach. The joint twist derivatives are adopted as degrees of freedom to assure inclusion of the strain due to simple twist. The u, v, and w displacement functions were proposed in Ref. 10. They are symbolized as follows:

$$\begin{aligned} u &= f_1'u_1 + f_2'u_2 + f_3'u_3 + f_4'u_4 = \lfloor f' \rfloor\{u\} \\ v &= f_1''v_1 + f_2''v_2 + f_3''v_3 + f_4''v_4 = \lfloor f'' \rfloor\{v\} \\ w &= f_1'''w_1 + f_2'''w_2 + \ldots + f_{16}'''\theta_{xy4} = \lfloor f''' \rfloor \begin{Bmatrix} w \\ \theta_x \\ \theta_y \\ \theta_{xy} \end{Bmatrix} \end{aligned} \tag{16}$$

The basic element linear stiffness matrix $[k]$ obtained as a consequence of the above assumed displacement functions have been presented in Ref. 10. When the matrix $[k]$ was evaluated in Ref. 10 by examples, rapid and monotonic convergence of the results to the analytic solutions with the refinement of gridwork was indicated. Explicit formulations for the coefficients in the matrix $[k]$ can be found in Ref. 10. The element incremental stiffness matrices $[n_0]$, $\frac{1}{2}[n_1]$, and $\frac{1}{3}[n_2]$ are derivable from Eq. (10). The matrices $\frac{1}{2}[n_1]$ and $\frac{1}{3}[n_2]$ can be found in Refs. 11 and 12. Thus the formulations are not to be repeated here.

III. Reduction Procedure

In view of the expense and limited computer capacity when applied to large-order (>100 degrees of freedom) systems, it is useful to have methods available for the reduction of the order of such systems prior to the solution of nonlinear matrix equations. Exact reduction procedure cannot be defined for the form of equations encountered in the present development; the method sought is therefore approximate in form. A corresponding form of the problem is met in the vibrational frequency analysis with the use of consistent mass matrices. Guyan[13] presented an approximate method of reducing the order of the latter. The method was then applied to the buckling problem to reduce the incremental stiffness matrices by Gallagher and the author.[14] In Ref. 14, the method was evaluated numerically and shown to be highly accurate.

The removal of certain degrees of freedom from the problem statement can be regarded as a transformation of coordinates from the complete set of degrees of freedom to the designated reduced number. This transformation of coordinates is derived from consideration of the linear stiffness matrix in isolation and it is suitable as an approximate transformation of the incremental stiffness matrix.

Consider first the analysis of a system excluding the nonlinear effects. The formulation is partitioned to distinguish between the degrees of freedom to be removed (subscript e) and those to be retained (subscript f). The former are unloaded degrees of freedom. Thus,

$$[K]\{Q\} = \{P\} \tag{17}$$

or

$$\begin{bmatrix} K_{ff} & K_{fe} \\ K_{ef} & K_{ee} \end{bmatrix} \begin{Bmatrix} Q_f \\ Q_e \end{Bmatrix} = \begin{Bmatrix} P_f \\ 0 \end{Bmatrix} \tag{18}$$

Solving for the displacement vector $\{Q_e\}$ in the lower equation of the partition and insertion of the result in the upper equation yields

$$[K_{ff} - K_{fe}K_{ee}^{-1}K_{ef}]\{Q_f\} = \{P_f\} \tag{19}$$

which can also be written in the form of a conventional transformation of coordinates:

$$[[T]^T[K][T]]\{Q_f\} = \{P_f\} \tag{20}$$

where $[T]$ is defined by

$$\begin{Bmatrix} Q_f \\ Q_e \end{Bmatrix} = \begin{bmatrix} [I] \\ -K_{ee}^{-1}K_{ef} \end{bmatrix}\{Q_f\} = [T]\{Q_f\} \tag{21}$$

In accordance with the view advanced above, the nonlinear formulation with an original form of

$$[[K] + [N]]\{Q\} = \{P\} \tag{22}$$

can be transformed into a reduced form as

$$[[T]^T[K][T] - [T]^T[N][T]]\{Q_f\} = \{P_f\} \tag{23}$$

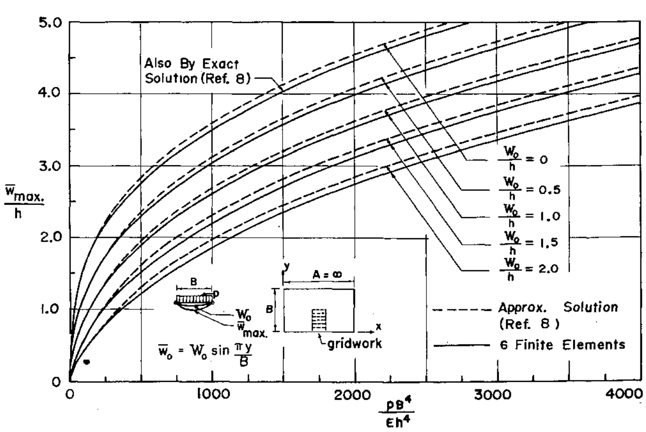

Fig. 2 Maximum deflections of long rectangular plates with all edges simply supported ($\nu = 0.3$, $A/B = \infty$).

in which the indicated matrices are partitioned with reference to the degrees of freedom "e" and "f."

The foregoing places no limitation on the selection of respective groups of degrees of freedom, other than the requirement that the system $\{Q_f\}$ correspond to stable support of the structure if it were to be suppressed. The illustrative examples of the following section will be performed by the use of this reduction scheme.

IV. Illustrative Examples

The applicability of the present nonlinear stiffness formulations for a plate finite element with initial deflection to the prediction of large deflection behavior of plate structures is demonstrated through a variety of examples.

The examples are performed by a step-by-step linear incremental procedure as described by Eq. (15). The non-dimensionalized loading level at the end of the nth step is suggested by a geometric series

$$\frac{pB^4}{Eh^4} = \sum_{i=1}^{n} c \cdot r^{i-1} \tag{24}$$

According to this equation, each load increment is greater than the previous one by r times. The load increment at ith step is specified by cr^{i-1}. For the class of examples examined in this section, the load-deflection curves show relatively high rate of change of slope at the beginning stage of loading. The rate decreases with the increase of loading level. This behavior means that smaller load increments are needed at the begining of loading and gradually larger load increments may be used at the higher loading levels. Eq. (24) appears to be one convenient way to suit this need, provided that the geometric constant r is properly chosen. In the following examples, the values of r are mostly chosen as 1.15 and 1.2.

Rectangular Long Plates

An appropriate basis of comparison for large deflection behavior for the initially deflected rectangular plate is the approximate solution given in Chap. 2 of Ref. 8. Vol'mir uses a single sinusoidal function to represent the deflection shape of an initially deflected long plate under uniform loads. The edges are assumed to be simply supported with no inplane movement. His results for the dimensionless pressure pB^4/Eh^4 vs maximum deflection $\bar{w}_{\max.}/h$ are shown in Fig. 2 for different values of initial deflections. The initial deflection is defined by

$$\bar{w}_o = W_o \sin(\pi y/B) \tag{25}$$

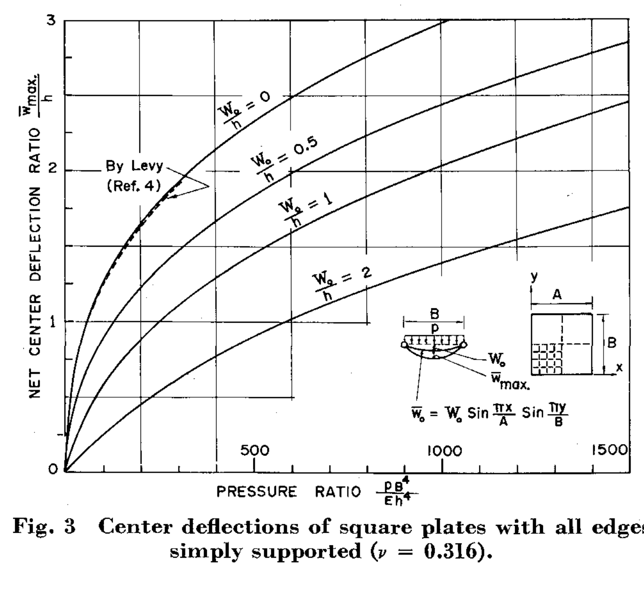

Fig. 3 Center deflections of square plates with all edges simply supported (ν = 0.316).

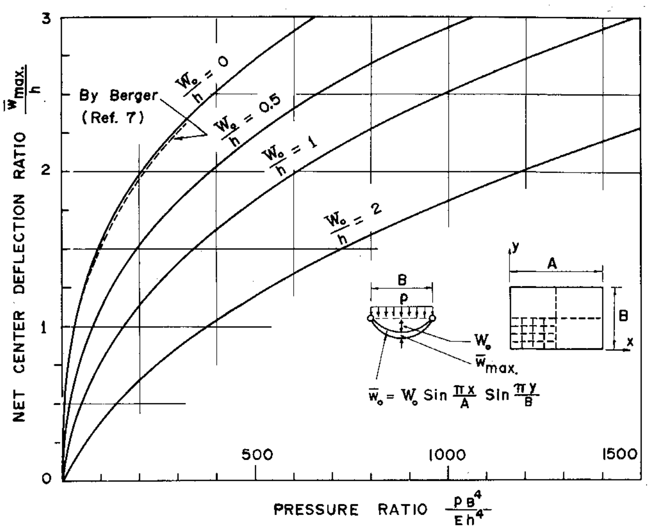

Fig. 4 Center deflections of rectangular plates with all edges simply supported (ν = 0.316, A/B = 1.5).

In the same reference, an exact solution for the same plate with no initial deflection given by Bubnov is also provided in Chap. 2.

Because of the symmetrical nature of the deflection shape, only a half of a strip across the plate width need be analyzed in the present finite element method. The half strip is idealized by 6 elements as shown in Fig. 2. In the incremental procedure, the load increment sizes are suggested by Eq. (24). The constants c and r are chosen to have the following values: a) for the cases $W_o/h = 0$ and 0.5, $c = 6$ and $r = 1.15$; b) for the cases $W_o/h = 1$–2, $c = 8$ and $r = 1.15$. The results are also shown in Fig. 2. For the particular case of no initial deflection ($W_o = 0$), the present results agree perfectly with the exact solution but show slight discrepancies from the approximate solution. For different values of initial deflection, $W_o = 0.5h$, h, $1.5h$, and $2h$, the discrepancies between the present results and the approximate solutions are consistent with those found in the case of no initial deflection.

Rectangular Plates with All Edges Simply Supported

For the uniformly loaded rectangular plates with all edges simply supported and with no inplane movement, two cases where the length-width ratios equal to 1 and 1.5 are examined. The initial deflection function is so assumed as to conform to the boundary conditions,

$$\bar{w}_o = W_o \sin(\pi x/A) \sin(\pi y/B) \tag{26}$$

Four cases where W_o have values 0, $0.5h$, h, and $2h$ are considered. Sixteen finite elements are used to idealize one quadrant of the plate (see Fig. 3). The constants c and r used in the linear incremental procedure are: a) for the case $A/B = 1$, 1.5 and $W_o/h = 0$, 0.5, $c = 6$ and $r = 1.15$; b) for the case $A/B = 1$, 1.5 and $W_o/h = 1$, 2, $c = 8$ and $r = 1.2$. The results for pressure vs center deflections are shown in Figs. 3 and 4. The alternative analytic solution for square plate with no initial deflection is available in Ref. 4 by Levy. He employed the double Fourier series to solve the exact governing equations of the plate by evaluating coefficients. Levy's solution is also shown in Fig. 3 and appears to be in very good agreement with the present results. For a rectangular plate with length-width ratio of 1.5 and with no initial deflection, the solution is available in Ref. 7 by Berger. He approximated the plate strain energy equation by neglecting the second strain invariant, $e_2 = \epsilon_x\epsilon_y - \frac{1}{4}\epsilon_{xy}^2$. He predicted the center deflection to a magnitude of $2.3h$. His results are also shown in Fig. 4 and appear to be in good agreement with the present solution.

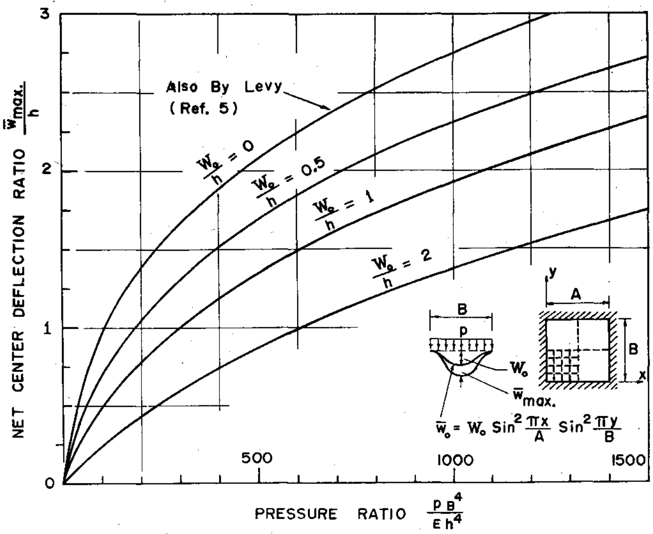

Fig. 5 Center deflections of square plates with all edges clamped (ν = 0.316).

Rectangular Plates with All Edges Clamped

For the uniformly loaded rectangular plate with all edges clamped with no inplane movement, two cases where the length-width ratios equal to 1 and 1.5 are examined. The initial deflection function is so assumed as to conform to the boundary conditions,

$$\bar{w}_o = W_o \sin^2(\pi x/A) \sin^2(\pi y/B) \qquad (27)$$

Four cases where W_o have values 0, 0.5h, h and 2h are considered. The same 16-element gridwork is used. The constants c and r chosen for the incremental procedure are: a) for the case $A/B = 1$ and $W_o/h = 0, 0.5$, $c = 20$ and $r = 1.15$; b) for the case $A/B = 1.0$ and $W_o/h = 1, 2$, $c = 25$ and $r = 1.15$; c) for the case $A/B = 1.5$ and $W_o/h = 0, 0.5$, $c = 12$ and $r = 1.15$; d) for the case $A/B = 1.5$ and $W_o/h = 1, 2$, $c = 16$ and $r = 1.2$. The results for pressure vs center deflection are shown in Figs. 5 and 6. The alternative analytic solutions for the plates with no initial deflection are available in Refs. 5 and 6. In both references, Levy and Greenman replaced the edge bending moment by equivalent pressure distribution and then applied the general Fourier series solution for the simply supported plate. Their results for center deflection are also shown in Figs. 5 and 6 for comparison. Two methods show almost identical results.

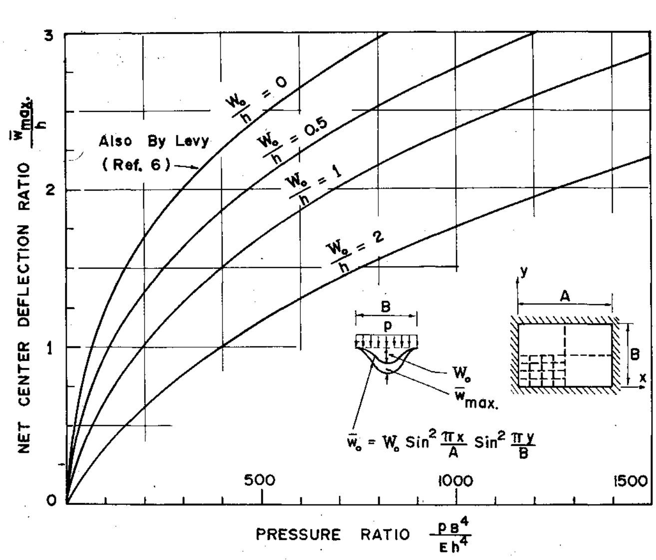

Fig. 6 Center deflections of rectangular plates with all edges clamped (ν = 0.316, A/B = 1.5).

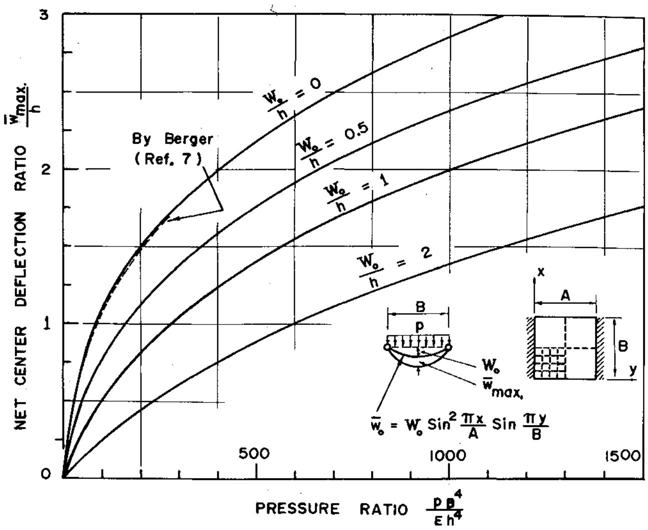

Fig. 7 Center deflections of square plates with two opposite edges simply supported and the other two clamped (ν = 0.316).

Rectangular Plates with Two Opposite Edges Simply Supported and the Other Two Clamped

For the uniformly loaded rectangular plate with two opposite edges simply supported and the other two clamped (no inplane movement), two cases where the ratios of the simply supported edge length A to clamped edge length B have values of 1 and 1.5 are examined. The initial deflection function is so assumed as to conform to the boundary conditions,

$$\bar{w}_o = W_o \sin^2(\pi x/A) \sin(\pi y/B) \qquad (28)$$

Four cases where W_o have values 0, 0.5h, h, and 2h are considered. The 16-element gridwork is used. The constants c and r chosen for the incremental procedure are: a) for the case $A/B = 1$ and $W_o/h = 0, 0.5$, $c = 12$ and $r = 1.15$; b) for the case $A/B = 1$ and $W_o/h = 1, 2$, $c = 16$ and $r = 1.15$; c) for the case $A/B = 1.5$ and $W_o/h = 0, 0.5$, $c = 6$ and $r = 1.15$; d) for the case $A/B = 1.5$ and $W_o/h = 1, 2$, $c = 8$ and $r = 1.2$.

The results for center deflection are shown in Figs. 7 and 8. The alternative analytic solutions for plates with no initial deflection are available in Ref. 7 by Berger. He approximated the plate strain energy equation by neglecting the

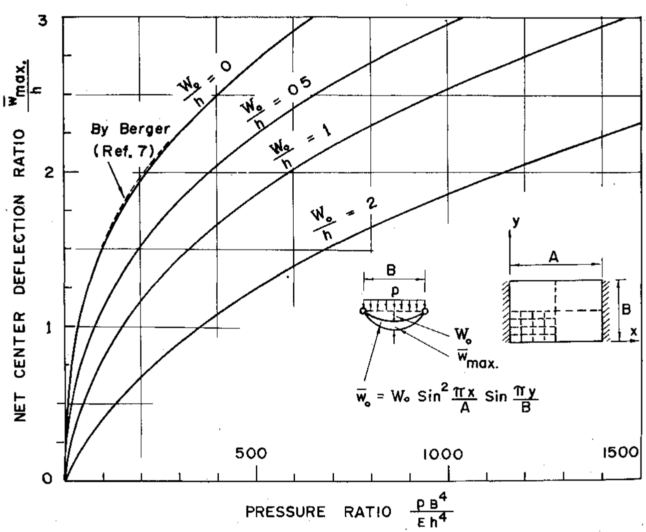

Fig. 8 Center deflections of rectangular plates with two opposite edges simply supported and the other two clamped ν = 0.316, A(S.S.)/B(Clamped) = 1.5).

second strain invariant. His approximate results are also shown in Figs. 7 and 8 for comparison. Good agreement is found.

V. Concluding Remarks

The formulations of nonlinear force-displacement matrix equations for initially deflected plate finite elements with large bending have been developed. The equations have been formulated explicitly for a conforming rectangular plate finite element, which can be used for either iterative or incremental approach. The formulations have been coded, on the basis of a linear incremental procedure, into an existing program for finite element nonlinear structural analysis, and have been evaluated through performance of analyses and comparisons of results with alternative analytic solutions.

It should be expected that the membrane and bending stresses evaluated by the present method will not be as accurate as the results shown for deflections. For one reason, these stresses are in terms of the derivatives of deflections. For the other, the gridwork must be further refined.

To select proper increment sizes is a difficult problem. Undersized increments are wasteful and oversized increments are inaccurate. The increment sizes should be based upon the rate of change of slope of the nonlinear behavior curves. In the present work, the load increment sizes are suggested by a geometric series as shown in Eq. (24). In the illustrative examples, efforts have been made to choose the values of geometric constant so that the increment sizes can agree with the rate of change of slope of the curves. In other words, the increment sizes are enlarged gradually as the rate of change of slope decreases. To achieve an optimum control of the increment sizes is a challenging problem which requires more studies.

References

[1] Gallagher, R. H. and Padlog, J., "Discrete Element Approach to Structural Instability Analysis," *AIAA Journal*, Vol. 1, No. 6, June 1963, pp. 1437–1438.

[2] Turner, M. J. et al., "Large Deflections of Structures Subjected to Heating and External Loads," *Journal of the Aerospace Sciences*, Vol. 27, No. 2, Feb. 1960, pp. 97–106.

[3] Mallett, R. H. and Marcal, P. V., "Finite Element Analysis of Nonlinear Structures," *Proceedings of the American Society of Civil Engineers, Journal of the Structural Division*, Vol. 94, No. ST9, Paper 6115, 1968, pp. 2081–2105.

[4] Levy, S., "Bending of Rectangular Plates with Large Deflections," TR 737, 1942, NACA.

[5] Levy, S., "Square Plate with Clamped Edges Under Normal Pressure Producing Large Deflections," TR 740, 1942, NACA.

[6] Levy, S. and Greenman, S., "Bending with Large Deflections of a Clamped Rectangular Plate with Length-Width Ratio of 1.5 Under Normal Pressure," TN 853, 1942, NACA.

[7] Berger, H. M., "A New Approach to the Analysis of Large Deflections of Plates," *Journal of Applied Mechanics*, Vol. 22, No. 4, Dec. 1955, pp. 465–472.

[8] Vol'mir, A. S., *Flexible Plates and Shells*, Translated from Russian, TR 66-216, 1967, Air Force Flight Dynamics Lab., Wright-Patterson Air Force Base, Ohio, pp. 13–14, 74–84.

[9] Timoshenko, S. P. and Woinowsky-Krieger, S., *Theory of Plates and Shells*, McGraw-Hill, New York, 1959, pp. 47 and 424.

[10] Bogner, F. K., Fox, R. L., and Schmit, L. A., "The Generation of Interelement-Compatible Stiffness and Mass Matrices by the Use of Interpolation Formulas," *Proceedings of the Conference on Matrix Methods in Structural Mechanics*, TR 66-80, Air Force Flight Dynamics Lab., Wright-Patterson Air Force Base, Ohio, 1966.

[11] Yang, T. Y., "Finite-Displacement Plate Flexure by Use of a Matrix Incremental Approach," *International Journal for Numerical Methods in Engineering to be published.*

[12] Yang, T. Y., "Flexible Plate Finite Element on Elastic Foundation," *Proceedings of the American Society of Civil Engineers, Journal of the Structural Division*, Vol. 96, No. ST10, Paper 7604, 1970.

[13] Guyan, R. J., "Reduction of Stiffness and Mass Matrices," *AIAA Journal*, Vol. 3, No. 2, Feb. 1965, p. 380.

[14] Gallagher, R. H. and Yang, T. Y., "Elastic Instability Predictions for Doubly Curved Shells," *Proceedings of the 2nd Conference on Matrix Methods in Structural Mechanics*, TR 68-150, 1969, Air Force Flight Dynamics Lab., Wright-Patterson Air Force Base, Ohio.